· claim : $\operatorname{ch} M(\Lambda) = e(\Lambda) \prod_{\alpha\in\Delta_+} (1-e(-\alpha))^{-\operatorname{mult}\alpha}$.

proof: let $\beta_1, \beta_2, \cdots$, be all the positive roots of the lie alg $g(A)$. and let $e_{-\beta_s, i_s}$ be a basis of $g_{-\beta_s}$ $(1 \le i_s \le \operatorname{mult}\beta_s = m_s)$

let v_Λ be a highest-weight vector of $M(\Lambda)$, by prop 9.2 b) we have $M(\Lambda) = U(n_-)v_\Lambda$ and $M(\Lambda) = \bigoplus_{\lambda\in H^*} (M(\Lambda))_\lambda$.

Then by PBW Th the vectors :

$e_{-\beta_1,1}^{n_{1,1}} \cdots e_{-\beta_1,m_1}^{n_{1,m_1}} e_{-\beta_2,1}^{n_{2,1}} \cdots e_{-\beta_2,m_2}^{n_{2,m_2}} \cdots (v_\Lambda)$ form a basis of $M(\Lambda)_\lambda$.

where $\underbrace{(n_{1,1}+\cdots n_{1,m_1})}_{r_1}\beta_1 + \underbrace{(n_{2,1}+\cdots+n_{2,m_2})}_{r_2}\beta_2 + \cdots = \Lambda - \lambda$.

Therefore $\operatorname{ch} M(\Lambda) = \sum_{\lambda\in H^*} (\dim (M(\Lambda))_\lambda) e(\lambda)$.

$$= \sum_{r_i} e(\Lambda - r_1\beta_1 - r_2\beta_2 - \cdots).$$

$$= e(\Lambda) \sum_{r_i} e(-r_1\beta_1 - r_2\beta_2 - \cdots) = e(\Lambda) \prod_{\beta_i} \left(\sum_{r_i} e(-r_i\beta_i)\right)$$

$$= e(\Lambda) \prod_{\beta_i} \left(-\sum_{j=1}^{m_i} e n_{ij}\beta_i\right) = e(\Lambda) \prod_{\beta_i} (e(0) + e(-\beta_i) + \cdots)^{m_i}$$

$$= e(\Lambda) \prod_{\beta_i} (1 - e(-\beta_i))^{-m_i}$$

$(1-e^{-\alpha})^{-1} = (1 + e^{\alpha} + e^{-2\alpha} + \cdots)$.

in f.d, $m_i = 1$. $\operatorname{ch} M(\Lambda) = e(\Lambda) \prod_{\beta_i} (1 - e(-\beta_i))$.

Q: in prop 9.8. (c_{ij})

find λ, $|\lambda+\rho|^2 = |\Lambda+\rho|^2$ $\lambda-\delta, \lambda-2\delta, \cdots$

$$(c_{ij}) = \begin{pmatrix} 1 & c_{12} & c_{13} & \\ & 1 & c_{23} & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

$$\begin{pmatrix} M(\lambda-\delta) \\ M(\lambda-2\delta) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & c_{12} & c_{13} & \\ & 1 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} L(\lambda-\delta) \\ L(\lambda-2\delta) \\ \vdots \end{pmatrix}$$

求左逆

重数与 λ 无关

$$\begin{pmatrix} 1 & a_{12} & \cdots \\ & \ddots & \\ & & \end{pmatrix} \begin{pmatrix} 1 & c_{12} & c_{13} \\ & 1 & \\ & & \ddots \end{pmatrix}$$

$c_{12} + a_{12} = 0.$

Q: in finite dimension Lie algebra.

$ch\,V$ is defined in the group ring of Λ over $\mathbb{Z}$, denoted by $\mathbb{Z}[\Lambda]$, where Λ is weight set $\Lambda = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Phi\}$.

in this book.

$ch\,V \in \mathcal{E}$, the element of $\mathcal{E}$ are seriese of the form

$\sum_{\lambda \in \mathfrak{h}^*} c_\lambda e(\lambda)$. $e(\lambda)\lambda = 1$ $e(\lambda)(\mu) = 0$ $\mu \neq \lambda$.

where $c_\lambda \in \mathbb{C}$, and $c_\lambda = 0$ for $\lambda \notin \bigcup_{i=1}^{s} D(\lambda_i)$.

§9.9. irreducibility and complete reducibility.

Now, using the Casimir operator, we can investigate irreducibility and complete reducibility in $\mathcal{O}$.

prop 9.9. let A be a symmetrizable matrix.

a) if $2(\Lambda + \rho \mid \beta) \neq (\beta \mid \beta)$ for every $\beta \in Q_+$, $\beta \neq 0$, then the $\mathfrak{g}(A)$-module $M(\Lambda)$ is irreducible.

proof: If $M(\Lambda)$ is not irreducible.

by prop 9.3 b). there is a primitive weight $\lambda = \Lambda - \beta$, where $\beta > 0$. Note that $[M(\Lambda) : L(\lambda)] \neq 0$ iff λ is a primitive weight. by lem 9.8 a). we have:

$$\underline{|\lambda+\rho|^2-|\rho|^2} = \underline{|\lambda+\beta|^2-|\beta|^2}$$

$\Rightarrow$ $\underline{|\lambda+\rho-\beta|^2 = |\lambda+\rho|^2}$ $\Rightarrow$ $2(\lambda+\rho|\beta) = (\beta|\beta)$. Contradiction

#.

b). If V is a $g(A)$-module from the category $\mathcal{O}$ such that for any two primitive weights λ and μ of V. such that $\lambda-\mu=\beta>0$. one has $2(\lambda+\rho|\beta) \neq (\beta|\beta)$. then V is complete reducible.

proof: we may assume that the $g(A)$-module V is indecomposable.

Since clearly, Ω is locally finite on V, i.e. every $v\in V$ lies in <u>a finite dimensional Ω-invariant subspace</u>

[$v\in V$, $v\in V_\lambda$, for some $\lambda\in P(V)$. 有限的原因.

$\dim(V_\lambda)<\infty$ & $\underline{\Omega(V_\lambda)\subset V_\lambda}$.

$\Omega.V_\lambda$ $h.(\Omega.V_\lambda) = \Omega(h.V_\lambda)$.].

· we obtain that there exists $a\in\mathbb{C}$ such that $\Omega-aI$ is locally nilpotent on V, i.e. $\forall v\in V$, there is $N\in\mathbb{Z}^+$, s.t. $(\Omega-aI)^N(v)=0$.

[Since V is a direct sum of 根子空间 ~~eigenspaces~~ of Ω.

$\Omega-aI$ acts locally nilpotently on V].

· Jordan 标准型. 每个 Jordan 块有限阶.

· Now, let v be a primitive vector of weight λ, then there is a submodule U s.t. $\Omega(v)=(|\lambda+\rho|^2-|\rho|^2)v \pmod U$ lem 9.8 b). then $|\lambda+\rho|^2-|\rho|^2=a$, whence $|\lambda+\rho|^2=|\mu+\rho|^2$ for any two primitive weight λ and μ.

Thus. $|\lambda+\rho|^2 = |\lambda-\beta+\rho|^2 = |\lambda+\rho|^2 - 2(\lambda+\rho|\beta) - (\beta|\beta)$

$\Rightarrow 2(\lambda+\rho|\beta) = (\beta|\beta)$ contradictos to condition.

by lem 9.5, we finish.

#.

§ 9.10.

· Recall: $g(A) = g'(A) + \mathcal{H}$ and that $\mathcal{H}' = \sum_i \mathbb{C}\alpha_i^\vee = g'(A) \cap \mathcal{H}$.

the free abelain group Q (root lattice) $Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$.

and $U(g'(A)) = \bigoplus_{\alpha\in Q} U'_\alpha$ where $U'_\alpha = U(g'(A)) \cap U_\alpha$, and

$U(g(A)) = \bigoplus_{\alpha\in Q} U_\alpha$, where $U_\alpha = \{x \in U(g(A)) \mid [h,x] = \alpha(h)x$, for $\forall h\in\mathcal{H}\}$.

· A $g'(A)$-module V is called a highest-weight module with highest weight $\Lambda \in (\mathcal{H}')^*$ if $V = \bigoplus_{\alpha\in Q_+} V_{\Lambda-\alpha}$ s.t.

(1) $U'_\beta(V_{\Lambda-\alpha}) \subset V_{\Lambda-\alpha+\beta}$.

(2) $\dim V_\Lambda = 1$.

(3) $h(v) = \lambda(h)v$, for $h \in (\mathcal{H}')^*$, $v \in V_\lambda$.

(4) $U(g'(A))(V_\lambda) = V$.

· In other words, this is a restriction of a highest-weight module over $g(A)$ to $g'(A)$.

· we define the Verma module $M(\Lambda)$ over $g'(A)$ and it contains a unique proper maximal graded submodule $M'(\Lambda)$.

· we put $L(\Lambda) = M(\Lambda)/M'(\Lambda)$

Lemma 9.10. The $g'(A)$-module $L(\Lambda)$ is irreducible.

proof: let $V \subset L(\Lambda)$ be a nonzero $g'(A)$-submodule.

we choose $v=\sum_{i=1}^{m} v_i \in V$ s.t. $v_i \in L(\Lambda)_{\lambda_i}$, $v_i \neq 0$ &

$\sum_i ht(\Lambda-\lambda_i)$ is minimal.

if $\lambda_i \neq \Lambda$ for some i, then $e_j(v_{\lambda_i}) \neq 0$ for some j.

then $e_j(v_i)$ will s.t. $\sum_i ht(\Lambda-\lambda_i)$ lower

Hence $v \in L(\Lambda)_\Lambda$, and $V=L(\Lambda)$

#

$g'(A)$ 部分 h 多一些 , $g(A)$ 少一些 E 和 F 相同.

$H \setminus H'$. $\langle \Lambda, \alpha_i^\vee \rangle$ $H' = g'(A) \cap \underset{\sim 中的}{H}$.

Claim : $\dim L(\Lambda) = 1$ iff $\Lambda|_{H'} = 0$.

proof: "$\Leftarrow$" if $\Lambda|_{H'}=0$. we can consider the 1-dimensional $g(A)$-module $\mathbb{C}$ which is trivial on $g'(A)$.

$n_+(1)=0$, $n_-(1)=0$, $h(1)=\langle\Lambda, h\rangle 1$. for $h \in H$. *

---

by the uniqueness of $L(\Lambda)$, then

$L(\Lambda) \cong \mathbb{C}$ as $g(A)$-module.

$\Rightarrow \dim L(\Lambda)=1$.

"$\Rightarrow$" since $\dim L(\Lambda)=1$. $\forall\, v \in L(\Lambda)$. is a highest-weight vector, then $0 = e_i f_i(v) = ([e_i, f_i] + \underline{f_i e_i})(v)$

$= \alpha_i^\vee \cdot v = \langle \Lambda, \alpha_i^\vee \rangle v$.

$\Rightarrow \Lambda|_{H'}=0$.

#.

prop 9.10.

let A be a symmetrisable matrix. possibly infinite.

a) if $2\langle \Lambda+\rho, \nu^{-1}(\beta)\rangle \neq (\beta|\beta)$ for every $\beta \in Q_+ \setminus \{0\}$. then the $g'(A)$-module $M(\Lambda)$ is irreducible.

b). Let V be $g'(A)$-module such that the following conditions are satisfied:

i) for every $v \in V$, $e_i(v) = 0$, for all but a finite number of the e_i (i.e. V is restricted).

ii) for every $v \in V$, there exists $k > 0$, s.t.

$e_{i_1} \cdots e_{i_s}(v) = 0$, whenever $s > k$. (权值也有限).

iii) $V = \bigoplus_{\lambda \in (H')^*} V_\lambda$, where $V_\lambda = \{v \in V \mid h(v) = \langle \lambda, h \rangle v$ for all $h \in H'\}$.

iv) if λ and $\mu \in (H')^*$ are primitive weights s.t. $\lambda - \mu = \beta|_{H'}$ for some $\beta \in Q_+ \setminus \{0\}$, then

$$2\langle \lambda + \rho, \nu^{-1}(\beta) \rangle \neq (\beta|\beta).$$

Then V is completely reducible, i.e. is isomorphic to a direct sum of $g'(A)$-modules of the $L(\lambda)$, $\lambda \in (H')^*$

proof: To prove the prop. we employ the operator Ω_0 instead of Ω.

[Pr: $\Omega_0 = 2 \sum_{\alpha \in \Delta_+} \sum_i e_{-\alpha}^{(i)} e_\alpha^{(i)}$, where $\{e_\alpha^{(i)}\}$ is a basis of g_α for $\forall \alpha \in \Delta_+$ and $\{e_{-\alpha}^{(i)}\}$ is the dual basis of $g_{-\alpha}$]

· Let is clear that Ω_0 is locally finite on V as long as condition (i) and (ii) of b) hold.

W-D. + (iii). V_λ f.d. $\Rightarrow$ locally finite.

$\Omega_0(v)$ 有限维.

· Furthermore. (2.6.1) $[\Omega_0, u] = -u(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha))$.

where $u \in U'_\alpha$, implies the following fact:

Let $v \in V_\lambda$ be s.t. $(\Omega_0 - aI_V)^k v = 0$. for some $k \in \mathbb{Z}_+$

and $a \in \mathbb{C}$. and let $v' \in U'_{-\beta}(v)$, $(\beta \in Q)$. then by (3.4.1). we have (9.10.2).

$$(\Omega_0 - (a + \underbrace{2\langle \lambda+\rho, \nu^{-1}(\beta)\rangle - (\beta|\beta)}_{b})I_v)^k v' = 0.$$

· ~~$a=0$ and $\Omega_0^k(v) = 0$ by (iii).~~

Ω_0 linear transformation. a 特征值.

(3.4.1) : $(x - \lambda - \mu)^k a = \sum_{s=0}^{k} C_k^s ((\mathrm{ad}\,x - \lambda)^s a)(x - \mu)^{k-s}$

where $k \geq 0$, $\lambda, \mu \in \mathbb{C}$.

$$(\Omega_0 - a - b)^k v' = \sum_{s=0}^{k} C_k^s ((\mathrm{ad}\,\Omega_0 - b)^s v')(\Omega_0 - aI_v)^{k-s}$$

since $v' \in U'_{\beta}(v)$. let $v' = uv$ where $u \in U'_{-\beta}$

then $\mathrm{ad}\,\Omega_0(u) = -u(2(\rho|-\beta) + (\beta|\beta) - 2\nu^{-1}(\beta))$.

we have $(\mathrm{ad}\,\Omega_0 - bI_v)v' = (\mathrm{ad}\,\Omega_0 - bI_v)uv$.

$$= \mathrm{ad}\,\Omega_0(u)v - bv'$$

$$= u(\underbrace{2(\rho|\beta) - (\beta|\beta)} + 2\nu^{-1}(\beta))v - 2\langle \lambda, \overbrace{\nu^{-1}(\beta)\rangle}^{(\lambda|\beta)} v' - 2\langle \underbrace{\rho, \nu^{-1}(\beta)}_{(\rho|\beta)}\rangle v' + (\beta|\beta)v'$$

$$= \cancel{(2(\rho|\beta) - (\beta|\beta))v'} + u\cdot 2\nu^{-1}(\beta)\cdot v - 2(\lambda|\beta)v' - \cancel{2(\rho|\beta)v'} \cancel{+(\beta|\beta)v'}$$

$$= 2u\langle \lambda, \nu^{-1}(\beta)\rangle v - 2\langle \lambda, \nu^{-1}(\beta)\rangle v' = 0.$$

$$\Rightarrow (\Omega_0 - (a + 2\langle \lambda+\rho, \nu^{-1}(\beta)\rangle - (\beta|\beta))I_v)^k v' = 0.$$